



FUNDAMENTALS OF ACOUSTICS

POLITECNICO DI MILANO –
CORSO DI MUSIC ENGINEERING

DIMOSTRAZIONI
(ordinate per argomento)

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OSCILLATORS

Active and Reactive force in a Harmonic oscillator.

Consider a harmonic oscillator driven by a force $F(t) = F_0 \cos(\omega t)$ and oscillating with a velocity

$$v(t) = v_0 \cos(\omega t + \theta)$$

- Demonstrate that the power delivered by the force can be decomposed into an active and a reactive part and explicitly write their expressions.
- Briefly describe their physical meaning.

The power P delivered by a force $F(t)$ acting on a particle is given by $P = F(t) \cdot v(t)$.

In this case, we have that $\left\{ \begin{array}{l} F(t) = F_0 \cdot \cos(\omega t) \\ v(t) = v_0 \cos(\omega t + \theta) \end{array} \right\}$ ^{ω is the angular frequency = $2\pi f$ where f is the frequency of the harmonic motion} so we have that $P = F_0 \cdot \cos(\omega t) \cdot v_0 \cos(\omega t + \theta) = F_0 \cdot v_0 \cos(\omega t) \cdot \cos(\omega t + \theta)$ ^{θ is the phase: it represents how the velocity is shifted compared to the force}

We can now use the trigonometric identity $\cos(A) \cdot \cos(B) = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$ into the previous power equation

$$\begin{aligned} P &= \frac{1}{2} F_0 \cdot v_0 [\cos(\omega t - \omega t - \theta) + \cos(\omega t + \omega t + \theta)] \\ &= \frac{1}{2} F_0 \cdot v_0 [\cos(\theta) + \cos(2\omega t + \theta)] \\ &= \frac{1}{2} F_0 \cdot v_0 \cos(\theta) + \frac{1}{2} F_0 \cdot v_0 \cos(2\omega t + \theta) \end{aligned}$$

Active power: represents the component of the force that is in phase with the velocity of the oscillator. This part of the force contributes directly to the oscillatory motion and does useful work on the system.

Reactive power: is the force that is 90 degrees out of phase with the velocity. This component doesn't directly contribute to the steady state motion.

Simple oscillator differential equation.

Write the one-dimensional differential equation for the linear motion of a mass M subject to an elastic restoring force (spring coefficient k). Derive analytically the expression for its resonance frequency.

- In order to get the equation that describes the one dimensional motion of a mass subject to an elastic restoring force we start from three known laws:
 - Newton's second law: the force applied on a mass is equal to $F = m \cdot a$. Since on the mass is applied only the elastic force, we have that $F = F_s$.
 - Hooke's law: the restoring force provided by the spring is $F_s = -k \cdot x$.

$$\begin{aligned} m \cdot a &= -k \cdot x \rightarrow m \cdot \frac{d^2 x}{dt^2} = -k \cdot x \xrightarrow{\text{divide all by } m} \frac{d^2 x}{dt^2} = -\frac{k}{m} \cdot x \rightarrow \frac{d^2 x}{dt^2} + \frac{k}{m} \cdot x = 0 \end{aligned}$$

- We know that the angular frequency $\omega = \sqrt{\frac{k}{m}}$ and that the relation between the frequency and the angular frequency is $\omega = 2\pi f$, so we can get $f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$.

Note: the resonance frequency is the oscillation frequency of the system in absence of any driving or damping force. It describes the frequency at which the system naturally prefers to oscillate. Operating a system near its resonance frequency can lead to large oscillations, which might be undesirable (musical instruments) or dangerous (bridges or buildings during earthquakes).

Damped oscillator

Problem 1 (7 points)

Write the differential equation for the motion of a mass M subject to a elastic restoring force (spring coefficient k) and a viscous damping (damping coefficient β). Demonstrate that the amplitude of its free oscillations is exponentially damped over time.

We know that $\left\{ \begin{array}{l} \text{the elastic force is } F_s = -kx \\ \text{the friction force is } F_f = -\beta v \\ \text{for the second Newton's Law it then that } -kx - \beta v = m \cdot a \end{array} \right\}$ we can solve the differential equation guessing that the solution is $b e^{\gamma t}$.

$$\begin{aligned} -kx - \beta \dot{x} &= m \ddot{x} \rightarrow -k b e^{\gamma t} - \beta b \gamma e^{\gamma t} = m b \gamma^2 e^{\gamma t} \rightarrow -k - \beta \gamma = m \gamma^2 \rightarrow m \gamma^2 + \beta \gamma + k = 0 \rightarrow \gamma = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m} = \lambda \pm j\omega \\ &= \frac{-\beta}{2m} \pm j \sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}} \omega \end{aligned}$$

Now we compute γ in the original differential equation:

$$x = b \cdot e^{(\lambda \pm j\omega)t} = b \cdot e^{\lambda t} \cdot e^{\pm j\omega t} \rightarrow x = b \cdot e^{\lambda t} \cdot (2 \cdot \cos(\omega t)) \rightarrow x = 2b e^{\lambda t} \cos(\omega t)$$

Equation of forced damped harmonic system impedance

Problem 1 (7 points)

Consider a harmonic oscillator of mass $M = 100$ g subject to an elastic restoring force (spring coefficient $k = 10$ N/m) and a viscous damping (damping coefficient $\beta = 0.1$ kg s⁻¹).

Derive the expression for the mechanical impedance of the resonator and calculate its value for a frequency of 10 Hz.

[Hint: recall that the mechanical impedance for a resonator subject to an external force F is defined as $Z = F/v$, where v is the velocity of the mass]

- We know that $\left\{ \begin{array}{l} \text{the elastic force is } F_e = -kx \\ \text{the friction force is } F_f = -\beta v \\ \text{for the second Newton's Law it has that } -kx - \beta v + F = m \cdot a \end{array} \right\}$ we want to express the equation in function of the velocity v .
 $-kx - \beta v + F = m \cdot a \rightarrow -kx - \beta v + F = m \cdot \ddot{x}$
- We know that the solution is in the form $x = b e^{j\omega t}$, and so $\left\{ \begin{array}{l} v = j\omega b e^{j\omega t} \\ a = (j\omega)^2 b e^{j\omega t} \end{array} \right\}$. We can see that $\left\{ \begin{array}{l} \text{perform the derivative = multiply by } j\omega \\ \text{perform the integral = divide by } j\omega \end{array} \right\}$
- We can write the differential equation in function of the velocity as follow.

$$-kx - \beta v + F = m \cdot \ddot{x} \rightarrow -k \frac{v}{j\omega} - \beta v + F = m \cdot j\omega v \rightarrow F = v \left(\frac{k}{j\omega} + \beta + m j\omega \right) \rightarrow \frac{F}{v} = \frac{k}{j\omega} + \beta + m j\omega = \text{impedance } Z$$

STRINGS

● Differential equation describing wave propagation in a one-dimensional infinite string.

Derive the differential equation describing wave propagation in a one-dimensional infinite string of linear mass density μ subject to a tension T . Calculate the propagation velocity for a wave on a string characterized by $T = 35 \text{ N}$ and $\mu = 2 \text{ g/m}$.

Consider a single infinitesimal element of the string. On this element are applied two forces: F_1 (one that goes up) $F_1 = (T \cdot \sin(\theta))_x \cdot dx$ and F_2 (one that goes down) $F_2 = (T \cdot \sin(\theta))_x \cdot dx$.

- We can approximate F_1 through the Taylor expansion, obtaining $F_1 = (T \cdot \sin(\theta))_x + \frac{\partial(T \cdot \sin(\theta))}{\partial x} dx$.

If we now calculate the net force $dF_y = F_1 - F_2$ we obtain $dF_y = \frac{\partial(T \cdot \sin(\theta))}{\partial x} dx$.

If we also consider θ very small, we have that $\sin(\theta) \approx \theta$ and $\theta \approx \frac{\partial y}{\partial x}$, so we get that $dF_y = \frac{\partial(T \cdot \theta)}{\partial x} dx = T \cdot \frac{\partial(\frac{\partial y}{\partial x})}{\partial x} dx \rightarrow dF_y = T \cdot \frac{\partial^2 y}{\partial x^2} dx$.

- Taking in account the Newton's second law ($F = m \cdot a$), we obtain the system:
$$\left. \begin{aligned} dF_y &= T \cdot \frac{\partial^2 y}{\partial x^2} dx \\ dF_y &= dm \cdot a_y = (\mu dx) \cdot \frac{\partial^2 y}{\partial t^2} \end{aligned} \right\} \text{from which we can extract the differential equation of a wave propagation in a one dimensional infinite string:}$$

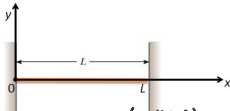
$$T \cdot \frac{\partial^2 y}{\partial x^2} dx = (\mu dx) \cdot \frac{\partial^2 y}{\partial t^2} \rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} \xrightarrow{\text{for small values of } y \rightarrow \text{divide}} \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

c: velocity of the wave over the string

Both ends fixed string's wave equation and frequency

Problem 2 (8 points)

Consider a string of linear mass density μ subject to a tension T . The string has a length L and is rigidly fixed at the two extremities. Starting from the general solution of the wave equation for the string and considering the reference frame in the figure, apply the proper boundary conditions to derive the expression for the resonance frequencies of the system.



The general solution of the wave equation is $\tilde{y}(x,t) = A \cdot e^{i(\omega t - kx + \phi_1)} + B \cdot e^{i(\omega t + kx + \phi_2)}$. We write it in exponential notation for convenience, but we are interested only in its real part.

$$y(x,t) = \text{Re}[\tilde{y}(x,t)] = A \cos(\omega t - kx + \phi_1) + B \cos(\omega t + kx + \phi_2) = A \cos(\omega t - kx) + B \sin(\omega t - kx) + C \cos(\omega t + kx) + D \sin(\omega t + kx)$$

$A \cos(\omega t - kx + \phi_1) = B \cos(\omega t - kx) + C \sin(\omega t - kx)$

The boundaries conditions for a finite string fixed at both ends are the following:

$$y(0,t) = 0 \quad A \cos(\omega t) + B \sin(\omega t) + C \cos(\omega t) + D \sin(\omega t) = (A + C) \cos(\omega t) + (B + D) \sin(\omega t) = 0$$

This condition in $x=0$ should be true for all the values of t , so we can chose the more convenient values of ωt in order to get the four constants.

$$\omega t = 0 \rightarrow (A + C) \cos(\omega t) + (B + D) \sin(\omega t) = 0 \rightarrow (A + C) \cos(0) = 0 \rightarrow A + C = 0$$

$\sin(0) = 0$

$$\omega t = \frac{\pi}{2} \rightarrow (A + C) \cos(\omega t) + (B + D) \sin(\omega t) = 0 \rightarrow (B + D) \sin(\omega t) = 0 \rightarrow B + D = 0$$

$\cos(\frac{\pi}{2}) = 0$

and so we can re-write the equation as $y(x,t) = A \cos(\omega t - kx) + B \sin(\omega t - kx) + C \cos(\omega t + kx) + D \sin(\omega t + kx)$

$$= A [\cos(\omega t - kx) - \cos(\omega t + kx)] + B [\sin(\omega t - kx) - \sin(\omega t + kx)]$$

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) \quad \sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) \quad \sin(\alpha - \beta) = \sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta)$$

$$= A [\cos(\omega t) \cos(kx) + \sin(\omega t) \sin(kx) - \cos(\omega t) \cos(kx) + \sin(\omega t) \sin(kx)] + B [\sin(\omega t) \cos(kx) - \cos(\omega t) \sin(kx) - \sin(\omega t) \cos(kx) - \cos(\omega t) \sin(kx)]$$

$$= A [2 \sin(\omega t) \sin(kx)] + B [-2 \cos(\omega t) \sin(kx)] = 2 [A \sin(\omega t) - B \cos(\omega t)] \sin(kx)$$

$$y(L,t) = 0 \quad 2 [A \sin(\omega t) - B \cos(\omega t)] \sin(kL) = 0 \rightarrow \sin(kL) = 0 \rightarrow kL = m \pi \rightarrow \frac{2\pi f}{\lambda} L = m \pi \rightarrow f = \frac{m v}{2L}$$

$\lambda = \frac{v}{f}, k = \frac{2\pi}{\lambda} = \frac{2\pi f}{v}$

Putting all together we get the solution for the both ends fixed strings: $y(x,t) = \sum_m [A_m \sin(\omega_m t) - B_m \cos(\omega_m t)] \sin(\frac{m\pi}{L} x)$

WAVES PROPAGATION

Most general equation for wave propagation

Consider a wave propagating along the x direction with velocity v .

- Demonstrate that its most general expression takes the form $f(x \pm vt)$.
- Discuss the meaning of the $+$ and $-$ sign.
- Verify that the wave function satisfies a one-dimensional wave equation.

- ④ We want to demonstrate that every wave that propagates in a specific direction (that we call x) can be expressed in the form $y = f(x - vt)$.
- We have in consideration a string on which a wave propagates and we assume that it's true that the equation $y = f(x - vt)$ describes our situation.
 - Then, at the time t_1 we take in consideration an arbitrary point x_1 of our string. The value of x_1 is given by the equation $y_1 = f(x_1 - vt_1)$.
 - We wait and, after a time interval t , the wave has propagated by a space vt . At this time about t_2 , we'll have a point x_2 that is x_1 shifted by vt . The point x_2 is described by the equation:

$$y_2 = f(x_2 - vt_2) = f(x_1 + vt - t_1 - vt_2) = f(x_1 - vt_1) = y_1 \text{ value of } x_2$$

$x_2 = x_1 + v(t_2 - t_1)$

The value of x_1 of t_1 is equal to the value x_2 of t_2 because every point on the wave profile moves to the right. This means that if we take the initial wave profile $f(x)$ and substitute it at $x = vt$, we get the wave profile at a later time t .

- ⑤ The $\pm$ sign in the argument of the wave function indicates the direction of the propagation of the wave. $\rightarrow f(x - vt)$, positive x direction
 $\rightarrow f(x + vt)$, negative x direction

- ⑥ The differential one-dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$. If we perform the second time derivatives and the second space derivatives of $f(x \pm vt)$ we can see that they verify the differential equation.

$$y = f(x_1 \pm vt_2) \quad \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2}{\partial t^2} f(x_1 \pm vt_2) = v^2 \frac{\partial^2}{\partial x^2} f(x_1 \pm vt_2) = v^2 \frac{\partial^2}{\partial x^2} y$$

differential equation verified.

Spherical waves equation, intensity and conservation of energy

Consider a harmonic spherical pressure wave propagating away from a point source (outwards).

- Write its most general analytical expression.
- Calculate the dependence of the intensity I of the spherical wave on the distance r from the source. Discuss why such a dependence is compatible with the conservation of energy.
- Given the Laplacian operator in spherical coordinates

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \right] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial}{\partial \theta} \right] + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

verify that the expression written in a) satisfies the three-dimensional wave equation.

- ① The most general spherical waves equation is $P(r, t) = \frac{A}{r} \cdot f(r - ct)$.

- ② For a spherical pressure wave emanating from a point source, the general solution in the far field is $p(r, t) = \frac{A}{r} \cos(kr - \omega t + \phi_0)$.
- The instantaneous intensity is the flux of energy per unit area. In order to calculate the instantaneous intensity equation we must:

- calculate the instantaneous energy $w(r, t) = \frac{P(r, t)^2}{2\rho c}$.

ρ = density of the medium
 c = speed of sound at the medium

- get the instantaneous intensity or flux of the instantaneous energy per unit area. $I(r, t) = c \cdot w(r, t) = \frac{P(r, t)^2}{2\rho c}$

the mean intensity over a period of the wave is obtained by averaging the instantaneous intensity over time.

$$p(r, t) = \frac{A}{r} \cos(kr - \omega t + \phi_0) \text{ and the average of } \cos^2(x) = \frac{1}{2}, \text{ so } \overline{p(r, t)} = \frac{A^2}{2r^2}$$

$$I(r, t) = \frac{P(r, t)^2}{2\rho c} \rightarrow \overline{I(r, t)} = \frac{\overline{A^2}}{2\rho c} = \frac{A^2}{2r^2 \rho c}$$

From this last equation we can see that the intensity decreases with the inverse of r^2 . This law conserves energy because the spherical area over which the energy is distributed increases proportionally to r^2 , and the same amount of energy is distributed over a bigger surface, giving a smaller intensity (that decreases proportionally to $1/r^2$).

- ③ We want to demonstrate that $P(r, t) = \frac{A}{r} \cdot f(r - ct)$ verifies the 3D-waves differential equation $\nabla^2 P = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2}$. Let's replace P with its equation and see what happens:

- $\nabla^2 P = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial P}{\partial r} \right] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial P}{\partial \theta} \right] + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 P}{\partial \phi^2}$. The spherical waves do not depend on θ and ϕ , so we can write that

$$\nabla^2 P = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial P}{\partial r} \right] \rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \left(\frac{A}{r} \cdot f(r - ct) \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \cdot \left(-\frac{A}{r^2} \cdot f(r - ct) + \frac{A}{r} \cdot f'(r - ct) \right) \right]$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \cdot \left(-\frac{A}{r^2} \cdot f(r - ct) + \frac{A}{r} \cdot f'(r - ct) \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} \left[-A \cdot f(r - ct) + A \cdot r \cdot f'(r - ct) \right]$$

$$= \frac{1}{r^2} \left(-f'(r - ct) + f'(r - ct) + r \cdot f''(r - ct) \right) = \frac{1}{r} \cdot f''(r - ct)$$

$$\nabla^2 P = \frac{1}{r} \cdot f''(r - ct) \rightarrow \frac{1}{r} \cdot f''(r - ct) = \frac{1}{c^2} \cdot \frac{1}{r} \cdot f''(r - ct) \rightarrow \text{identity verified.}$$

WAVES IN FLUIDS

Differential equation of the force balance in fluids.

Demonstrate, by properly justifying all passages, that the force balance for a fluid volume confined between the planes located at x and $x+dx$ can be written in the form of a differential equation as

$$-\frac{\partial p}{\partial x} = \rho_{TOT} \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \right)$$

where ρ_{TOT} is the mass density, p is the pressure, and v_x is the velocity of the fluid.

For a fluid element of thickness dx and cross-sectional area S we have that:
 { the pressure force on the left face is $P(x,t)$
 the pressure force on the right face is $P(x+dx,t)$.

- We are considering an infinitesimal volume, so the distance between the two volume's faces is infinitesimal and we can approximate the pressure curve as x (displacement) as a straight line. In this case we can express the difference of pressure between the two faces as $\frac{\partial p}{\partial x} dx$. We can write the following identity

$$\underbrace{P(x+dx,t)}_{\text{pressure on the right face}} = \underbrace{P(x,t)}_{\text{pressure on the left face}} + \underbrace{\frac{\partial P}{\partial x} dx}_{\text{difference of pressure}} \rightarrow P(x+dx,t) - P(x,t) = \frac{\partial P}{\partial x} dx \rightarrow -P(x+dx,t) + P(x,t) = -\frac{\partial P}{\partial x} dx \rightarrow \underbrace{S \left[-P(x+dx,t) + P(x,t) \right]}_{= S \cdot \Delta P = dF_x = \text{difference of force applied by the two faces}} = -S \left[\frac{\partial P}{\partial x} dx \right]$$

So we have obtained that $dF_x = -S \left[\frac{\partial P}{\partial x} dx \right]$. We can calculate dF_x also with the second Newton law $F = m \cdot a$, where $m = \rho \cdot dV = \rho \cdot S dx$ (calculation $a = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x}$)

With these informations, we can write the following system:

$$\begin{cases} dF_x = -S \left[\frac{\partial P}{\partial x} dx \right] \\ dF_x = \rho_{TOT} S dx \left(\frac{dv_x}{dt} + v_x \frac{\partial v_x}{\partial x} \right) \end{cases} \rightarrow S \left[\frac{\partial P}{\partial x} dx \right] = \rho_{TOT} S dx \cdot \left(\frac{dv_x}{dt} + v_x \frac{\partial v_x}{\partial x} \right) \rightarrow \rho \left[\frac{\partial P}{\partial x} dx \right] = \rho_{TOT} \left(\frac{dv_x}{dt} + v_x \frac{\partial v_x}{\partial x} \right) \rightarrow \frac{\partial P}{\partial x} = \rho_{TOT} \left(\frac{dv_x}{dt} + v_x \frac{\partial v_x}{\partial x} \right)$$

Note: This velocity has a component that depends on the time and the other that depends on the space, so we have to perform a material total derivative.

Differential equation of the mass balance in fluids.

Demonstrate that the mass balance for a fluid volume confined between the planes located at x and $x+dx$ can be written in the form of a differential equation as

$$-v_x \frac{\partial \rho_{TOT}}{\partial x} - \rho_{TOT} \frac{\partial v_x}{\partial x} = \frac{\partial \rho}{\partial t}$$

where ρ_{TOT} is the total mass density, ρ is the mass density variation due to wave propagation, and v_x is the velocity of the fluid.

Note: $\rho_{TOT} = \rho + \rho_{static}$ where ρ_{static} is the density of fluid at equilibrium, meanwhile ρ is the variable part of the density that changes due to the wave propagation.

We want to express the variation of mass through a surface over time in a unit of time dt . We know that $\begin{cases} dm = \rho \cdot dV \\ dV = S \cdot v_x \cdot dt \end{cases}$ (where ρ is the density, S is the surface, and v_x is the velocity).

- With the previous informations we can write that $\frac{dm}{dt} = \rho_{TOT} (S \cdot v_x) = \rho_{TOT} S v_x$
- Now we want to write the mass influx during the time dt : $dm_{(x)} - dm_{(x+dx)} = \rho_{TOT} S v_x(x,t) S - \rho_{TOT} S v_x(x+dx,t) S$
- We can apply the Taylor expansion on the $\rho_{TOT} S v_x(x+dx,t) S$, obtaining that $\rho_{TOT} S v_x(x+dx,t) S = \left[\rho_{TOT} S v_x(x,t) + \frac{\partial \rho_{TOT} S v_x(x,t)}{\partial x} dx \right] S$
- Replacing the term in the net influx expression, we obtain:

$$dm_{(x)} - dm_{(x+dx)} = \rho_{TOT} S v_x(x,t) S - \left[\rho_{TOT} S v_x(x,t) + \frac{\partial \rho_{TOT} S v_x(x,t)}{\partial x} dx \right] S = -\frac{\partial \rho_{TOT} S v_x(x,t)}{\partial x} dx S$$

- The net mass influx through the surface of the volume must be equal to the variation of mass inside the volume, so we can write that:

$$\frac{\partial \rho_{TOT} S v_x(x,t)}{\partial x} dx S = S \cdot dx \cdot \frac{\partial \rho}{\partial t} \rightarrow \frac{\partial \rho_{TOT} S v_x(x,t)}{\partial x} \cdot \frac{dx}{dx} \cdot S = S \cdot \frac{\partial \rho}{\partial t} \rightarrow \frac{\partial \rho_{TOT} S v_x(x,t)}{\partial x} = \frac{\partial \rho}{\partial t}$$

where dV is the variation of density over time.

$$\downarrow v_x \cdot \frac{\partial \rho_{TOT}}{\partial x} + \rho_{TOT} \frac{\partial v_x}{\partial x} = \frac{\partial \rho}{\partial t}$$

One dimension (linear) second order wave equation.

Problem 3 (7 points)

Demonstrate that the force and mass balances for a fluid volume confined between the planes located at x and $x+\Delta x$

$$\begin{cases} -\frac{\partial p}{\partial x} = \rho_{TOT} \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \right) \\ -v_x \frac{\partial \rho_{TOT}}{\partial x} - \rho_{TOT} \frac{\partial v_x}{\partial x} = \frac{\partial \rho}{\partial t} \end{cases}$$

can be combined, after suitable assumptions, to give a one-dimensional second-order differential wave equation. State clearly all the assumptions and briefly discuss their validity.

- We can do the following assumptions:
 - the variations of pressure are very small and we can consider $P_{TOT} \approx P_0$
 - the variations of pressure are very small and we can consider $P_{TOT} \approx P_0$
 - the velocity is very small: $v_x \approx 0$
- Generally, the equation that describes the relation between the density and the pressure is in the form $f(P_{TOT}, P_{TOT}, T) = 0$, but if we assume an isentropic situation (no heat exchange) the temperature becomes a fixed value. This means that the relation assumes the shape $f(P_{TOT}, P_{TOT})$ and so we can assume a linear equation of state $P = c^2 \rho$, where c^2 is a positive constant.
- Considering these assumptions, we can re-write the system as follows:

$$\begin{aligned} -\frac{\partial P}{\partial x} &= \rho_0 \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \right) \rightarrow -\frac{\partial P}{\partial x} = \rho_0 \frac{\partial v_x}{\partial t} \quad \text{1-dimension full side} \rightarrow -\frac{\partial P}{\partial x} = \rho_0 \frac{\partial}{\partial x} \frac{\partial v_x}{\partial t} \\ -v_x \frac{\partial \rho_{TOT}}{\partial x} - \rho_{TOT} \frac{\partial v_x}{\partial x} &= \frac{\partial \rho}{\partial t} \rightarrow -\rho_0 \frac{\partial v_x}{\partial x} = \frac{\partial \rho}{\partial t} \rightarrow -\rho_0 \frac{\partial v_x}{\partial x} = \frac{\partial}{\partial t} \frac{P}{c^2} \rightarrow \rho_0 \frac{\partial v_x}{\partial x} = -\frac{1}{c^2} \frac{\partial P}{\partial t} \rightarrow \rho_0 \frac{\partial}{\partial x} \frac{\partial v_x}{\partial x} = -\frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \\ P &= c^2 \rho \rightarrow \rho = \frac{P}{c^2} \end{aligned}$$

Due to the density of the medium greater, these two terms are equivalent.

ρ is the variation of density caused by the volume.

- Equating the two last results we get $+\frac{\partial^2 P}{\partial x^2} = +\frac{1}{c^2} \frac{\partial^2 P}{\partial t^2}$, that is the differential linear wave equation.

Fluid velocity and impedance for two counter-propagating waves

Problem 2 (10 points)

Consider two counter-propagating pressure sound waves that can be written as

$$p(x, t) = A e^{j(\omega t - kx)} + B e^{j(\omega t + kx)}$$

and propagate in a fluid with average mass density ρ_0 .

- By using the proper differential equation, calculate the expression for the fluid velocity $v_x(x, t)$.
- Derive the values for the characteristic impedances of the two (forward and backward) waves.

- We know that the pressure P and the velocity correlation can be expressed by the differential equation $\frac{\partial P}{\partial x} = -\rho_0 \frac{\partial v_x}{\partial t}$, so we can calculate the velocity as follows:

$$\begin{aligned} P &= A e^{j(\omega t - kx)} + B e^{j(\omega t + kx)} \\ \frac{\partial P}{\partial x} &= -\rho_0 \frac{\partial v_x}{\partial t} \rightarrow -jk A e^{j(\omega t - kx)} + jk B e^{j(\omega t + kx)} = -\rho_0 \frac{\partial v_x}{\partial t} \rightarrow v_x = \frac{jk}{\omega} \left(A e^{j(\omega t - kx)} + B e^{j(\omega t + kx)} \right) \\ &= \frac{jk}{\omega} \left(A e^{j(\omega t - kx)} + B e^{j(\omega t + kx)} \right) \quad \text{1-dimension full side} \\ &= \frac{jk}{\omega} \left(A e^{j(\omega t - kx)} + B e^{j(\omega t + kx)} \right) \quad \text{1-dimension full side} \\ &= \frac{jk}{\omega} \left(A e^{j(\omega t - kx)} + B e^{j(\omega t + kx)} \right) \quad \text{1-dimension full side} \end{aligned}$$

- We can rewrite $\frac{k}{\omega} \rightarrow \frac{k}{\omega} = \frac{2\pi}{\lambda} \cdot \frac{1}{2\pi f} = \frac{1}{\lambda f} = \frac{1}{c}$ and so the equation becomes:

$$v_x = \frac{k}{\omega \rho_0} \left(A e^{j(\omega t - kx)} - B e^{j(\omega t + kx)} \right) = \frac{1}{c \rho_0} \left(A e^{j(\omega t - kx)} - B e^{j(\omega t + kx)} \right) = \frac{P_{(x,t)}}{Z_0} - \frac{P_{(x,t)}}{Z_0}$$

$c \cdot \rho_0 = Z_0$

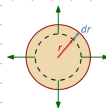
- The impedances Z_1 and Z_2 have the same absolute value $= Z_0$ but Z_2 has the $-$ sign because:
 - Z is defined as $\frac{P}{v}$
 - v is negative due to the waves direction.

POINT SOURCES

● Relation between $\hat{Q}$ and A. Intensity I as function of Q.

Consider a harmonic spherical pressure wave $p(r, t) = \frac{A}{r} e^{j(\omega t - kr)}$ propagating away from a monopole point source of volume velocity $Q(t) = Q_0 e^{j\omega t}$.

- Derive the relation between the amplitude Q_0 of the volume velocity and the amplitude A of the spherical wave.
- Calculate the time-averaged intensity of the spherical wave as a function of the volume velocity Q_0 .



- ① • The variation of volume of the point source is $dV = 4\pi r^2 dr$
- We can calculate the volume $Q \cdot dt$ $Q \cdot dt = 4\pi r^2 \cdot dr \rightarrow Q = 4\pi r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \cdot v_r$

We can express v_r or $v_r = \frac{p}{\rho_0 c} \cdot \left(1 + \frac{1}{jkr}\right)$ and we can write that $Q = 4\pi r^2 \cdot \frac{p}{\rho_0 c} \cdot \left(1 + \frac{1}{jkr}\right)$

We are considering a point source, so r is very small. If $r \rightarrow 0$ we can approximate the equation as follows:

$$\lim_{r \rightarrow 0} Q = \lim_{r \rightarrow 0} 4\pi r^2 \cdot \frac{p}{\rho_0 c} \cdot \left(1 + \frac{1}{jkr}\right) \rightarrow Q = 4\pi r^2 \cdot \frac{p}{\rho_0 c} \cdot \frac{1}{jkr} = \frac{4\pi r p}{\rho_0 \omega \cdot j}$$

this term becomes negligible

$c = \lambda \cdot f = \frac{2\pi}{k} \cdot f = \frac{2\pi}{k} \cdot \frac{\omega}{2\pi} = \frac{\omega}{k} \rightarrow c \cdot k = \omega$

- The pressure of a spherical wave is defined by the equation $P = \frac{A}{r} \cdot e^{j(\omega t - kr)}$ we can ignore the propagation factor

$$Q = \frac{4\pi r p}{\rho_0 \omega \cdot j} = \frac{4\pi r}{\rho_0 \omega \cdot j} \cdot \frac{A}{r} \cdot e^{j(\omega t - kr)} = \frac{4\pi A}{\rho_0 \omega \cdot j} \cdot e^{j(\omega t - kr)} = \hat{Q} \cdot e^{j(\omega t)}$$

$\frac{4\pi A}{\rho_0 \omega \cdot j} = \hat{Q}$ (amplitude of the volume velocity)

- The equation $\frac{4\pi A}{\rho_0 \omega \cdot j} \cdot e^{j(\omega t - kr)} = \hat{Q} \cdot e^{j(\omega t)}$ that we just found shows the relation between A and $\hat{Q}$

- ② In order to get the averaged time intensity, we need the pressure P. From the passage above, we can take the following identity:

$$Q = \frac{4\pi r p}{\rho_0 \omega \cdot j} = \hat{Q} \cdot e^{j(\omega t)} \rightarrow p = \frac{\rho_0 \omega \cdot j}{4\pi r} \cdot \hat{Q} \cdot e^{j(\omega t)}$$

Putting the just found relation into the intensity formula $I = \frac{A p_{\text{rms}}^2}{2 \cdot Z_c}$ we get that:

$$I = \frac{A p_{\text{rms}}^2}{2 \cdot Z_c} = \frac{1}{2 \rho_0 c} \left| \frac{\rho_0 \omega \cdot j}{4\pi r} \cdot \hat{Q} \cdot e^{j(\omega t)} \right|_{\text{max}}^2 = \frac{1}{2 \rho_0 c} \left| \left(\frac{\rho_0 \omega \cdot j}{4\pi r} \right) \right|^2 = \frac{1}{2 \rho_0 c} \left| \left(\frac{\rho_0^2 \omega^2 \cdot 1}{16 \pi^2 r^2} \right) \right| = \frac{\rho_0 \omega^2 \hat{Q}^2}{32 \cdot c \pi^2 r^2}$$

$Z_c = \rho_0 c$

INTERFERENCE

Amplitude modulator: factor when two time-periodic signals interfere.

Problem 1 (4 points)

Consider two time-periodic signals possessing the following expression in complex notation, where A is a real number:

$$S_1(t) = A e^{j(\omega - \Delta\omega)t}$$

$$S_2(t) = A e^{j(\omega + \Delta\omega)t}$$

Demonstrate analytically that when the two signals interfere their amplitude is modulated as $\cos(\Delta\omega t)$.

When $S_1(t)$ and $S_2(t)$ interfere, the resulting field is $S(t) = S_1(t) + S_2(t) = A \cdot e^{j(\omega - \Delta\omega)t} + A \cdot e^{j(\omega + \Delta\omega)t} = A \cdot (e^{-j\Delta\omega t} + e^{j\Delta\omega t}) e^{j\omega t}$

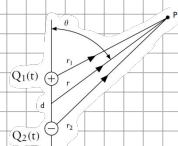
$$S(t) = 2A \cos(\Delta\omega t) e^{j\omega t} \leftarrow A \cdot (2 \cos(\Delta\omega t)) e^{j\omega t}$$

Equation of pressure in the far field scattering from the volume velocity Q .

Problem 3 (11 points)

Consider a dipolar source made of two point sources at distance d with volume velocities $Q_{1,2} = \pm Q_0 e^{j\omega t}$ (Q_0 being a real number):

- Derive the approximate expression for the spherical pressure wave $p(r, t)$ generated in the far field ($r \gg d$).
- For $\lambda = d$, find the angular position of the maxima and minima in the radiation pattern (please clarify with a drawing the direction of the angular position $\theta = 0$ for reference).



- Given $Q_1 = Q_0 e^{j\omega t}$ and $Q_2 = -Q_0 e^{j\omega t}$ and knowing that for a single sound source the pressure $P = \frac{\rho_0 \omega \cdot j}{4\pi r} \cdot Q \cdot e^{j(\omega t + kr)}$ we can write that the total pressure wave generated by the two point sources is equal to $P = \frac{\rho_0 \omega \cdot j}{4\pi r_1} \cdot Q_0 e^{j(\omega t + kr_1)} - \frac{\rho_0 \omega \cdot j}{4\pi r_2} \cdot Q_0 e^{j(\omega t + kr_2)}$
- We are operating in the far field, so $r_1 \approx R + \frac{d}{2} \cdot \cos(\theta)$ ($r_2 \approx R - \frac{d}{2} \cdot \cos(\theta)$). We can rewrite the equation as follow.

$$P = \frac{\rho_0 \omega \cdot j}{4\pi R} \cdot (Q_0 e^{j(kr_1)} - Q_0 e^{j(kr_2)}) e^{j(\omega t)} = \frac{\rho_0 \omega \cdot j}{4\pi R} \cdot (Q_0 e^{j(k[R + \frac{d}{2} \cdot \cos(\theta)])} - Q_0 e^{j(k[R - \frac{d}{2} \cdot \cos(\theta)])}) e^{j(\omega t)} =$$

you need to not consider the difference between r_1 and r_2 .

$$= \frac{\rho_0 \omega \cdot Q_0}{4\pi R} \cdot (e^{j(k \frac{d}{2} \cdot \cos(\theta))} - e^{-j(k \frac{d}{2} \cdot \cos(\theta))}) e^{j(\omega t + kR)} = \frac{\rho_0 \omega \cdot Q_0}{4\pi R} \cdot 2 \cdot \sin(k \frac{d}{2} \cdot \cos(\theta)) \cdot e^{j(\omega t + kR)}$$

$$= \frac{\rho_0 \omega \cdot Q_0}{2\pi R} \cdot \sin(k \frac{d}{2} \cdot \cos(\theta)) \cdot e^{j(\omega t + kR)}$$

$\sin(x) = \frac{1}{j} (e^{jx} - e^{-jx})$

- The maxima are located where $\sin(k \frac{d}{2} \cdot \cos(\theta)) = \pm 1 \rightarrow k \frac{d}{2} \cdot \cos(\theta) = \frac{\pi}{2} + m\pi \rightarrow k \frac{d}{2} \cdot \cos(\theta) = \frac{\pi}{2} + m\pi$
 - for $m=0 \rightarrow \theta = \pm \frac{\pi}{2}$
 - for $m=1 \rightarrow \theta = \pm \frac{3\pi}{2}$
- The minima are located where $\sin(k \frac{d}{2} \cdot \cos(\theta)) = 0 \rightarrow k \frac{d}{2} \cdot \cos(\theta) = m\pi \rightarrow k \frac{d}{2} \cdot \cos(\theta) = m\pi \rightarrow \cos(\theta) = m$
 - for $m=0 \rightarrow \theta = \pm \frac{\pi}{2}$
 - for $m=1 \rightarrow \theta = 0$
 - for $m=-1 \rightarrow \theta = \pi$



DOPPLER EFFECT

● Doppler effect (moving point source)

Problem 4 (6 points)

Consider a point source emitting a spherical sound wave of frequency f and moving away from an observer with velocity V_s (the observer is fixed in the reference frame of the fluid).

- Derive the expression for the frequency perceived by the observer.
- Is the perceived pitch higher or lower than the one perceived when the source is not moving away from the observer?

Ⓐ Ⓑ We usually calculate the wavelength as $\lambda = \frac{v}{f}$, where v is the speed of sound in the fluid (v_s) and f the frequency of the wave.

If the source moves, the speed of the wave changes:

- if the source is moving away from the observer, the wave velocity becomes $v = v_s + V_s$ and $\lambda' = \frac{v}{f} = \frac{v_s + V_s}{f}$ (It is a truth which the sound waves perceive away from the sound source)
- the speed of sound c is constant, $\Rightarrow v_s = f \cdot \lambda = f' \cdot \lambda' \Rightarrow v_s = f' \frac{v_s + V_s}{f} \Rightarrow f' = \frac{v_s \cdot f}{v_s + V_s} = \frac{f}{1 + \frac{V_s}{v_s}}$ (source velocity)
- We can see that in this case, the perceived wavelength is longer $\rightarrow$ the frequency is lower $\rightarrow$ the pitch is lower than the "no-moving source" case.
- if the source is moving through the observer, the wave velocity becomes $v = v_s - V_s$ and $\lambda' = \frac{v}{f} = \frac{v_s - V_s}{f}$ (source velocity)
- the speed of sound c is constant, $\Rightarrow v_s = f \cdot \lambda = f' \cdot \lambda' \Rightarrow v_s = f' \frac{v_s - V_s}{f} \Rightarrow f' = \frac{v_s \cdot f}{v_s - V_s} = \frac{f}{1 - \frac{V_s}{v_s}}$
- We can see that in this case, the perceived wavelength is shorter $\rightarrow$ the frequency is higher $\rightarrow$ the pitch is higher than the "no-moving source" case.

● Doppler effect (moving listener)

Problem 2 (5 points)

Consider a point source emitting a spherical sound wave of frequency f and an observer moving away from the source with velocity V_r (the source is fixed in the reference frame of the fluid).

- Derive the expression for the frequency perceived by the observer.
- Is the perceived pitch higher or lower than the one perceived when the observer is not moving away from the source?

Ⓐ Ⓑ We know the identity $c = \lambda \cdot f$, where c is the velocity of the sound wave. In the case of a moving listener, the perceived c is not just the speed of the sound c , but will increase/decrease following the listener velocity $\rightarrow$ it is a truth which the sound waves perceive away from the sound source.

If the listener goes through the sound source, he will perceive a higher frequency where $c = c_0 + V_r$ and so $c = \lambda \cdot f \Rightarrow c_0 + V_r \rightarrow f = \frac{c_0 + V_r}{\lambda} = f_0 \left(1 + \frac{V_r}{c_0}\right)$

In this case the listener will hear a higher frequency than the "original one" and so the pitch will be higher.

If the listener moves away from the point source, he will perceive a lower frequency where $c = c_0 - V_r$ and so $c = \lambda \cdot f \Rightarrow c_0 - V_r \rightarrow f = \frac{c_0 - V_r}{\lambda} = f_0 \left(1 - \frac{V_r}{c_0}\right)$

In this case the listener will hear a lower frequency than the "original one" and so the pitch will be lower.

PIPES

● Input impedance of a pipe.

Problem 4 (8 points)

Consider a pipe of length $L = 40$ cm, terminated with one open end and one rigid wall.

- Calculate the fundamental resonance frequency of the pipe.
- Determine the input impedance at the open end of the pipe as a function of frequency.

● Let's start considering on infinite transmission line = infinite pipe. We start from the following hypothesis:
 - we neglect attenuation
 - the sound waves in a plane wave
 - we consider impedance at it cross discrete.

● We must consider two counter propagating waves. In this scenario we have that

$$\begin{cases} P(x) = A e^{-jkx} + B e^{jkx} \\ Z_0 v(x) = A e^{-jkx} - B e^{jkx} \end{cases} \xrightarrow{\text{remembering that } e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)} \begin{cases} P(x) = A (\cos(kx) + j \sin(kx)) + B (\cos(kx) - j \sin(kx)) = \cos(kx) (A+B) - j \sin(kx) (A-B) \\ Z_0 v(x) = A (\cos(kx) + j \sin(kx)) - B (\cos(kx) - j \sin(kx)) = \cos(kx) (A-B) - j \sin(kx) (A+B) \end{cases}$$

● Now we can apply the first boundary condition, which is that for each point x , we have a pressure $P(x)$ and a velocity $v(x)$. Consider $\begin{cases} P(0) \\ Z_0 v(0) \end{cases}$:

$$\begin{cases} P(0) = \cos(0) (A+B) - j \sin(0) (A-B) = A+B \\ Z_0 v(0) = \cos(0) (A-B) - j \sin(0) (A+B) = A-B \end{cases} \rightarrow \begin{cases} P(0) = \cos(0) (A+B) - j \sin(0) (A-B) = A+B \\ Z_0 v(0) = \cos(0) (A-B) - j \sin(0) (A+B) = A-B \end{cases}$$

We just found that $A+B = P(0)$ and that $A-B = Z_0 v(0)$. Now we can put these result into the original equation:

$$\begin{cases} P(x) = \cos(kx) (P(0)) - j \sin(kx) (Z_0 v(0)) \\ Z_0 v(x) = \cos(kx) (Z_0 v(0)) - j \sin(kx) (P(0)) \end{cases} \xrightarrow{\text{divide the velocity because it will be useful in the next step}} v(x) = \frac{\cos(kx) (Z_0 v(0)) - j \sin(kx) (P(0))}{Z_0}$$

● The passage done until now are for an infinite transmission line. Now we can pass to a finite one, considering the following situation:

- Assume that in the point that we have previously defined as $x=0$ there is now the end wall of the tube with an impedance Z_L . We also assume that the open end of the tube is located at the coordinate $-L$.
- Knowing that $Z = \frac{P}{v}$ we can write the following.

$$\begin{aligned} Z_{in} &= \frac{P(-L)}{v(-L)} = \frac{\cos(kL) (P(0)) - j \sin(kL) (Z_0 v(0))}{\cos(kL) (Z_0 v(0)) - j \sin(kL) (P(0))} = Z_0 \frac{\cos(kL) (P(0)) - j \sin(kL) (Z_0 v(0))}{\cos(kL) (Z_0 v(0)) - j \sin(kL) (P(0))} \\ &= Z_0 \frac{\frac{\cos(kL) (P(0)) - j \sin(kL) (Z_0 v(0))}{\cos(kL) (Z_0 v(0)) - j \sin(kL) (P(0))}}{\frac{\cos(kL) (Z_0 v(0)) - j \sin(kL) (P(0))}{\cos(kL) (Z_0 v(0)) - j \sin(kL) (P(0))}} = Z_0 \frac{\frac{P(0)}{v(0)} - j \tan(kL) \cdot Z_0}{Z_0 - j \tan(kL) \cdot \frac{P(0)}{v(0)}} \rightarrow Z_{in} = Z_0 \frac{Z_L + j \tan(kL) \cdot Z_0}{Z_0 + j \tan(kL) \cdot Z_L} \end{aligned}$$

● If we want to express Z_{in} as a function of f , we can put $k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$ into the Z_{in} formula.

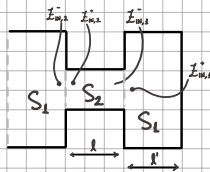
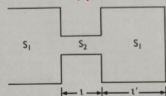
$$\lambda = \frac{c}{f}$$

if $Z_L \approx \infty$ we can approximate the equation
 or $Z_0 \frac{Z_L + j \tan(kL) \cdot Z_0}{Z_0 + j \tan(kL) \cdot Z_L} \approx Z_0 \frac{Z_L}{j \tan(kL) \cdot Z_L} = -j Z_0 \cot(kL)$

● Resonance frequency of the Helmholtz resonator (pipes with discontinuities)

Problem 2 (9 points)

Use the impedance description for the propagation of sound waves along a finite transmission line to derive the resonance frequency of the Helmholtz resonator sketched in the figure (where S_1 and S_2 are the surface areas of the two transverse sections, under the assumption that the lengths l and l' are much shorter than the wavelength of the sound).



We want to get the equation of the Helmholtz resonator's resonance frequency. We must proceed as follows:

- We can calculate $Z_{in,1}^+$ with the formula of the finite transmission line impedance: $Z_{in,1}^+ = -jZ_0 \cot(kl')$, and if we consider $kl' \ll 1$ we can approximate the tan with its content, writing that

$$Z_{in,1}^+ = jZ_0 \frac{1}{kl'} = -j \frac{\rho_0 c^2}{l'w} = \frac{\rho_0 c^2}{j l' w}$$

- We can calculate $Z_{in,1}^-$ with the formula of the impedance discontinuity $\frac{Z_1}{Z_2} = \frac{S_2}{S_1} \cdot \frac{Z_{in,2}^-}{Z_{in,1}^-} \rightarrow Z_{in,1}^- = \frac{S_2}{S_1} Z_{in,2}^-$

- We calculate $Z_{in,2}^+$ with the general formula of the input impedance for finite transmission line $Z_{in,2}^+ = Z_0 \frac{Z_L + jZ_0 \tan(kl)}{Z_0 + jZ_L \tan(kl)}$

$$Z_{in,2}^+ = Z_0 \cdot \frac{Z_{in,1}^- + jZ_0 \tan(kl)}{Z_0 + jZ_{in,1}^- \tan(kl)}$$

- We can calculate $Z_{in,2}^-$ with the formula of the impedance discontinuity $\frac{Z_1}{Z_2} = \frac{S_1}{S_2} \cdot \frac{Z_{in,2}^-}{Z_{in,1}^-} \rightarrow Z_{in,2}^- = \frac{S_1}{S_2} Z_{in,1}^-$

- Therefore we can calculate the resonance frequency or $\omega_{res} = \sqrt{\frac{c^2 S_2}{kl' S_1}}$

$\hookrightarrow$ S_1 is the volume of the two bigger sections.

